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APPLICATION OF $\bar{\partial}_b$ TO DEFORMATION OF ISOLATED SINGULARITIES *)

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§1. Introduction

The purpose of the present note is to give the outline of deformation theory of isolated singularities based on Tangential Cauchy Riemann Equations. The deformation theory of singularities is already developed by several mathematicians. (For a historical note see the article of O. Forster in this volume [1].) However the methods so far are algebraic. That is to say, deformations are regarded as deformations of defining equations of singularities. As for the analytic approach, besides the one exposed here, Richard Hamilton constructed a theory which relies on $\bar{\partial}$ operator on a tubular neighborhood of the boundary. This approach leads to a non-linear boundary value (non coercive) problem of Cauchy Riemann equations [2].

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§2. Isolated singularities and CR structures

Let V be a Stein analytic space (of complex dimension n) smooth except at a point p such that there is a relatively compact open neighborhood U of p in V with the property that the boundary of U is smooth and strongly pseudo convex. Denote by M the boundary of U. M being a real submanifold of codimension one in the complex manifold $V - \{p\}$, the complex tangent vector bundle $\underset{\sim}{C}TM$ has a distinguished sub-bundle ${}^{o}T''_{V}M$. Namely, it consists of all elements in $\underset{\sim}{C}TM$ which are of type $(0,1)$ when we regard $\underset{\sim}{C}TM$ as a sub-bundle of $\underset{\sim}{C}TV|M$. Setting $E'' = {}^{o}T''_{V}M$, the equation

$$(1) \qquad Xf = 0 \qquad \text{for all } X \in E''$$

is called the tangential Cauchy Riemann equation on M induced by the ambiant complex space V. By the construction it is obvious that the sub-bundle $E'' = {}^{o}T''_{V}M$ satisfies the following conditions:

(2) If L and L' are sections of E'', so is $[L,L']$.

We refer any sub-bundle E'' of $\underset{\sim}{C}TM$ of fiber dimension $n-1$ with the property $E' \cap E'' = \{0\}$, $E' = \overline{E''}$, as an almost CR structure on M. If it further satisfies the condition (2), we call E'' a CR structure. The notion of pseudo convexity of a hypersurface in a complex manifold can be formulated solely in terms of the almost CR structure induced in the hypersurface. Namely, pick sections $Y_1,\dots,Y_{n-1}$ of E'' on an open set G of M such that they generate $E''|G$ and also a real vector field T on G complementary to $E' + E''$. Write

$$[Y_j, \overline{Y}_k] \equiv c_{jk}\, iT \qquad (\mathrm{mod}\ Y_1,\dots,Y_{n-1},\overline{Y}_1,\dots,\overline{Y}_{n-1}) .$$

We say that E'' is strongly pseudo convex when the hermitian matrix (c_{jk}) obtained in this way is always non-singular and its eigenvalues are of the same sign. If M is strongly pseudo convex in V, ${}^{o}T''_{V}M$ is strongly pseudo convex. Conversely, for any strongly pseudo convex CR structure E'' on M and for any point p in M, Boutet de Monvel [6] showed recently that there are $f_1,\dots,f_n \in C^\infty(M)$ such that $Xf_j = 0$ for any section X of E'' and the map $x \to (f_1(x),\dots,f_n(x)) \in \underline{C}^n$ is an embedding on a neighborhood of p. This means in particular that any strongly pseudo convex CR-structure E'', when restricted to small open sets, is induced by an ambiant complex manifold. Pick an open covering $\{G_\alpha\}$ of M together with an ambiant complex manifold W_α of G_α such that $W_\alpha - G_\alpha$ consists of two components and $E''|G_\alpha$ is the induced CR-structure on G_α by W_α. Then by the theorem of H. Lewy [3] there is unique component W'_α

of $W_\alpha - G_\alpha$ such that any solution of the equation (1) on W_α extends uniquely to a holomorphic function on W'_α, provided we chose W_α sufficiently thin. This will allow you to piece together W'_α (shrinking G_α a little if necessary) and construct a complex manifold N with boundary M (regarding the pseudo concave part of the boundary of N as open), even though the complex structure may not extend beyond M. We may say that E'' is induced by the ambiant complex manifold N. Since $f_1,\ldots,f_n$ in the theorem of Boutet de Monvel are defined everywhere in M, we may conclude that we can construct N as above such that the holomorphic functions on N separate points. Then it is a theorem of H. Rossi [8] that we can fill in the hole on N. He showed that the set of the maximal ideals of the algebra of the holomorphic functions on N, say S, has the natural structure of normal Stein analytic space. The obvious injection $N \to S$ is holomorphic and the image is open. In this way we can replace the deformations of normal isolated singularities by the deformations of CR-structures.

Deformations of isolated singularities may be viewed in two steps. Namely, the first is the deformations of the smooth part of the analytic set and the second is the way singular points are added to complete it. Now the second step is not unique. Because of the blowing up and its inverse, this step is very complicated.

Our contention is that the CR-structure induced on the boundary completely controls the first step and also gives one definite way of doing the second step.

§3. Integrability conditions

We develop the deformation theory of CR-structures following the pattern established in the deformation theory of complex structures. Let us recall the first step of the latter. We fix a reference complex structure on a manifold, say N, which is considered as a sub-bundle T'' of the complex tangent vector bundle $\underline{C}TN$. T'' consists of complex tangent vectors of type $(0,1)$. We note the direct sum decomposition

(3) $$\underline{C}TN = T'' + T' , \quad T' = \overline{T}''.$$

Then any almost complex structure sufficiently close to T'' is considered as a sub-bundle which is a graph of a bundle map

(4) $$\omega : T'' \to T' .$$

We denote this almost complex structure by T''_ω. Thus almost complex structures sufficiently close to T'' are parameterized by T'- valued differential forms of type $(0,1)$. T'' has parameter 0. If T''_ω is a complex structure, it follows that

(5) $$\bar{\partial}\omega - \tfrac{1}{2}[\omega,\omega] = 0$$

where $[\omega,\omega]$ is the type $(0,2)$ form constructed by means of the bracket of vector fields (and exterior product. It is the famous theorem of Newlander and Nirenberg that the converse is true. These considerations are the gound on which we can apply the theory of elliptic differential operators to construct the versal family of deformations of compact complex structures.

As for the reference CR-structure ${}^{o}T''$, there is no canonical decomposition like (3). We are forced to choose one. We pick a sub-bundle F of $\underline{C}TM$ of fiber dimension 1 such that

(6) $$\underline{C}TM = {}^{o}T'' + {}^{o}T' + F \ , \quad \overline{{}^{o}T''} = {}^{o}T' \ , \quad \overline{F} = F \ .$$

Then any almost CR-structure sufficiently close to ${}^{o}T''$ is the graph of a bundle map

(7) $$\varphi' : {}^{o}T'' \to {}^{o}T' + F \ .$$

It is a little awkward to use a bundle like ${}^{o}T'' + F$ to parameterize almost CR-structures. We avoid this by observing that the restriction to $\underline{C}TM$ of the canonical projection map $CTV|M \to T'V|M$ has the kernel ${}^{o}T''$ and hence this map induces an isomorphism of ${}^{o}T' + F$ to $T'V|M$. Denote by

(8) $$\tau : T'V|M \to {}^{o}T' + F$$

the inverse of the isomorphism. Then we can write

(9) $$\varphi' = \tau \circ \varphi$$

where

(10) $$\varphi : {}^{o}T'' \to T'V|M$$

is a bundle map. Thus almost CR-structures on M sufficiently close to ${}^{o}T''$ are parameterized by $T'V|M$ valued differential forms of type $(0,1)_b$. Namely, ${}^{o}T''_{\varphi}$ is the graph of the bundle map $\tau \circ \varphi : T'V|M \to {}^{o}T' + F$. Therefore

(11) $${}^{o}T''_{\varphi} = \{X - \tau \circ \varphi(X);\ X \in {}^{o}T''\}.$$

In other words we have the isomorphism:

(11)' $${}^{o}T'' \ni X \to X - \tau \circ \varphi(X) \in {}^{o}T''_{\varphi}.$$

The next problem is to decide which of the almost CR-structures ${}^{o}T''_{\varphi}$ are CR-structures. This will lead to an equation like (5) for φ. Now we can rewrite the integrability condition (2) in terms of differential forms as follows:

(2)' If θ is a differential form of degree 1 such that $\theta(X) = 0$ for all $X \in E''$, then $d\theta(X,X') = 0$ for all $X, X' \in E''$.

Thus we can write down the condition for ${}^{o}T''_{\varphi}$ being a CR-strucute when we can find a generator for differential forms of degree 1 which annihilate ${}^{o}T''_{\varphi}$. To do this we use local chart and introduce several notations. Before proceeding, we pause here to note that the condition (2)' can be reformulated in the following more suggestive way: Consider the diagram

$$\text{(12)}\qquad \begin{array}{ccc} \Lambda^1(M,\, C) & \rightarrow & \Lambda^2(M,\, C') \\ \downarrow & & \downarrow \\ C^\infty(M,(E'')^*) & \dashrightarrow & C^\infty(M,\Lambda^2(E'')^*) \end{array}$$

where the vertical arrows are induced by the injection $E'' \rightarrow CTM$. Then the condition (2)' is equivalent with the following:

(2)" There is a unique dotted arrow which makes the diagram (11) commutative.

When E'' is a CR-structure, we denote by $\bar{\partial}_{E''}$ the dotted arrow obtained in (2)". It follows easily that once this can be done, we can construct similarly the differential operator

$$\text{(13)}\qquad \bar{\partial}_{E''} : C^\infty(M,\Lambda^p(E'')^*) \rightarrow C^\infty(M,\Lambda^{p+1}(E'')^*.$$

Since we always have the differential operator $\bar{\partial}_{E''} : C^\infty(M,\, C) \rightarrow C^\infty(M,(E'')^*)$ as the composition of the exterior derivative and the restriction map, we conclude that

E'' is a CR-structure if and only if we have the $\bar{\partial}_{E''}$- complex

$$(14) \qquad C^\infty(M, C) \to C^\infty(M, \Lambda^1(E'')^*) \to C^\infty(M, \Lambda^2(E'')^*) \to \dots \quad .$$

When $E'' = {}^oT''$, this is the $\bar{\partial}_b$- complex.

Now we come back to the problem of writing down the generator mentioned above. Let our reference CR-structure ${}^oT''$ be induced locally by a real submanifold G in an open ball in $\underline{C}^n$. Denote by $z = (z^1, \dots, z^n)$ the general elements in the ball, and let

$$(15) \qquad h = 0$$

be the equation of G , where h is a real valued C^∞ function in z . The choice of h is not unique. But we pick one and preserve it throughout. By the injection i of G into the ball in $\underline{C}^n$ we identify $\underline{C}TG$ as a sub-bundle of $\underline{C}T\underline{C}^n | G$. Since F in (6) is preserved under conjugation, we can write

$$(16) \qquad F = \underline{C}(P' - P'') \ , \quad P' \in T'\underline{C}^n | G \ , \quad P'' = \overline{P'} \ .$$

We normalize the choice of P' by the requirement

$$(17) \qquad \langle dh, P' \rangle = 1 \ .$$

Set

(18) $$d'h = \Sigma_k h_k dz^k , \quad h_{\bar{k}} = \bar{h}_k .$$

(19) $$P' = \Sigma_k p^k \partial/\partial z^k , \quad p^{\bar{k}} = \overline{p^k} .$$

By (17),

(17)' $$\Sigma_k p^k h_k = 1 .$$

Set

(20) $$z^{\bar{k}} = i^*(d\bar{z}^k - p^{\bar{k}} d''h) ,$$

(21) $$Z_{\bar{k}} = \partial/\partial \bar{z}^k - h_{\bar{k}} P'' .$$

By (17) we see easily that $Z_{\bar{k}} \in {}^oT''$. $z^{\bar{k}}$ generates all differential forms of degree one which annihilate ${}^oT' + F$. $Z_{\bar{k}}$ generates ${}^oT''$. They have the relations:

(22) $$\Sigma_k h_{\bar{k}} z^{\bar{k}} = 0 , \quad \Sigma_k p^{\bar{k}} Z_{\bar{k}} = 0 .$$

Now a $T'V|M$ valued differential form of type $(0,1)_b$, say φ , can be expressed on G as

(23) $$\varphi = \Sigma_k \varphi^k \otimes \frac{\partial}{\partial z^k} , \quad \varphi^k = \Sigma_\ell \varphi^k_{\bar{\ell}} z^{\bar{\ell}} \quad \text{with}$$

$$\Sigma p^{\bar{\ell}} \varphi^k_{\bar{\ell}} = 0 .$$

Because of (11), the diffential forms on G of degree 1 which annihilate ${}^{o}T''_{\varphi}$ are generated by

(24) $$\theta^k = i^*dz^k + \varphi^k \qquad (k = 1,\ldots,n)\ .$$

Since ${}^{o}T''_{\varphi}$ is a CR-structure (assuming that φ is sufficiently small) if and only if $E'' = {}^{o}T''_{\varphi}$ satisfies the condition (2)', it follows that ${}^{o}T''_{\varphi}$ is a CR-structure if and only if (since $d\theta^k = d\varphi^k$)

(25) $$d\varphi^k \equiv 0 \qquad (\mathrm{mod}\ \theta^1,\ldots,\theta^n)\ .$$

Set

(26) $$\partial^{\tau}/\partial z^k = \tau(\partial/\partial z^k) = \partial/\partial z^k - h_k P''\ .$$

We calculate the condition (25) more explicitly using the expression (23). Then we arrive at the following conclusion: For a sufficiently small $T'V|M$ valued differential form φ of type $(0,1)_b$, ${}^{o}T''_{\varphi}|G$ is a CR-structure if and only if

(27) $$P(\varphi) = \overline{\partial}_b\varphi - \Sigma_{j,k,\ell}(\partial^{\tau}\varphi^k_{\overline{\ell}}/\partial z^i)\varphi^i \wedge z^{\overline{\ell}} \otimes \partial/\partial z^k$$
$$+ (\Sigma_i h_i\varphi^i) \wedge \Sigma_{\ell,k}(\overline{\partial}_b p^{\overline{\ell}} - \Sigma_i \varphi^i \partial^{\tau} p^{\overline{\ell}}/\partial z^i)\varphi^k_{\overline{\ell}} \otimes \partial/\partial z^k$$

vanishes identically.

$P(\varphi)$ is constructed depending on the chart z of the ambiant complex manifold inducing ${}^{o}T''$ and of the function h

in (15). However one can show that $P(\varphi)$ is independent of such choice. This can be done by explicitly calculating the right hand side of (27) when we make changes in the choice. Recently D.C. Spencer and H. Goldshmidt found an intrinsic defining formula of $P(\varphi)$.

§4. Heuristic argument for the construction of versal families

Let us recall the basic idea in the construction of the versal families of deformations of compact complex manifolds.

A diffeomorphism of N transforms an almost complex structure to an almost complex structure. Thus the diffeomorphism group of N acts on the set of almost complex structures on N . This action sends complex structures to complex structures. Two structures on the same orbit are isomorphic structures. Since we are interested in deformations we consider only almost complex structures sufficiently close to the reference complex structure T'' and actions of diffeomorphisms sufficiently close to the identity map. Hence we may describe our situation roughly as follows: A sufficiently small open neighborhood of 0 in $\Lambda^{(0,1)}(N,T')$ is fibered into orbits by the action of small open neighborhoods of the identity in the diffeomorphism group of N . We consider the subset of this fiber space consisting of all ω such that ${}^{0}T''_{\omega}$ is a complex structure. This is a fiber subspace say B. If we can find a cross-section passing through 0, say C, of fibers of B, $\{T''_\omega ; \omega \in C$ will be considered as a universal family of deformations of N. However, it can happen (for some N) that it is impossible to find a decent such C. This is due to the fact that the dimension of the complex automorphism group of T''_ω which acts as the isotopy group at ω may change with ω. To avoid this difficulty we fiber B instead into orbits by action of diffeomorphisms which are complementary to the automorphism group of T''. To be more precise, we parameterize first a small neighborhood of the identity in the diffeomorphism group of N by a small neighborhood of 0 in $C^\infty(N,T')$ by an exponential map. For a small $\xi \in C^\infty(N,T')$ denote by g_ξ the diffeomorphism parameterized by ξ. Written in a complex chart z of N

$$(28) \qquad g_{t\xi}(z)^k \equiv z^k + t\xi^k \pmod{t^2}, \qquad \xi = \Sigma_k \xi^k \, \partial/\partial z^k .$$

Denote by ${}^\perp C^\infty(N,T')$ the subspace of $C^\infty(N,T')$ orthogonal to the subspace of holomorphic sections of T' (with respect to a hermitian metric in N). Let $\overset{\perp}{G}N$ be the set of diffeomorphisms of N parameterized by elements in a small neighborhood of 0 in ${}^\perp C^\infty(N,T')$. Now, instead of fibering B into orbits by small neighborhoods of the identity in the diffeomorphism group of N, we fiber B into orbits by small neighborhoods of the identity in $G^{\perp}N$. Then it is possible to find a cross-section. A family of defomrations of N constructed in this way is the versal family of deformations of N. Before we proceed further, we insert here a notation. For a small $\omega \in \Lambda^{(0,1)}(N,T')$ and a diffeomorphism g sufficiently close to identity map N, the transform of T''_{ω} by f is equal to T''_{θ}. We set $\theta = \omega \circ g$. Then we find that

$$(29) \qquad \omega \circ g_{\xi} = \omega + \bar{\partial}\xi + \dots$$

where ... includes all terms which are not linear in (ω, ξ). This formula plays an important role in the construction of the versal family.

Now we start to carry over the above consideration to deformations of isolated singularities viewed as deformations of CR-structures. Then we notice a new phenomenon due to the fact that we can wiggle the boundary. Let ${}^{o}T''_{\varphi}$ be a CR-structure on M induced by an ambiant complex manifold N_1. Since N_1 is diffeomorphic to the ambiant complex manifold N of ${}^{o}T''$, we may write $N_1 = N_{\omega}$. Let $f: M \to N$ be a C^{∞} injective map sufficiently close to the injection $i: M \to N$. The complex manifold N_{ω} induces a CR-structure on $f(M)$, which we transplant to a CR-structure on M via f. We call it the transform of ${}^{o}T''_{\varphi}$ by f. Since the above process is nothing but a wiggling of the boundary it is obvious that ${}^{o}T''_{\varphi}$ and its transform give rise to isomorphic singularities. Thus we find that in the deformation theory of isolated singularities the set of injections $M \to N$ sufficiently close to i plays the role of diffeomorphism group in the deformation theory of complex structures. This is the only modification we have to make.

Let us go over the fibering we consider more explicitly. We first parameterize injections of M into N sufficiently close to i by a small open neighborhood of $C^\infty(M,T'N|M)$, say f_ξ for $\xi \in C^\infty(M,T'NM)$, by an exponential map. In a complex analytic chart z of N it means that

$$(30) \qquad f_\xi^k(z) = z^k + \xi^k(z) + \dots , \qquad \xi = \Sigma\, \xi^k \partial/\partial z^k$$

for $z \in M$. Denote by I the set of all f_ξ where ξ are sufficiently small and orthogonal to the vector space of $\bar\partial_b$ closed sections of $T'N|M$. Consider the set B of φ such that φ is sufficiently small and ${}^oT''_\varphi$ is induced by a complex structure N_ω. The elements in I act on the elements in B. Consider the fibering of B into orbits of the action by small neighborhoods of i in I. We shall try to find a cross-section of the fiber space B and show that a cross-section represents a versal family of deformations of the isolated singularity out of which we obtained ${}^oT''$.

We might feel that our picture is a little blurred because we considered only φ such that ${}^{o}T''_{\varphi}$ is induced by a complex manifold N_{ω} which lies on both sides of M, whereas in §2 we constucted for any integrable φ a complex manifold which induces ${}^{o}T''_{\varphi}$ but lies only in one side of M. However this does not stop us from constructing the versal family by the following reason: To construct a family which we wish to be the versal family we do not need ω's , and in order to show that the family we constructed is versal we are offered to consider only φ's such that the ambiant complex manifolds lie on both sides of M. The last is due to the fact that we start from an analytic set V with an isolated singularity so that ${}^{o}T''$ is induced by an ambiant complex manifold which lies on both sides of M and that any small deformation of V induces a CR-structure with the same property.

As before we define $\varphi\circ f$ so that the transform of ${}^{o}T''_{\varphi}$ by f is ${}^{o}T''_{\varphi\circ f}$. Then we find after a little calculation by

$$\text{(31)} \qquad \varphi\circ f_{\xi} = \varphi + \bar{\partial}_b \xi + \dots$$

where ... includes all terms not linear in (ω, ξ).

§5. The construction of versal families

We recall first how the versal family of deformations of a compact complex manifold, say N, was constructed. As was explained in the preceding section, we are to find a decent set C of $\omega \in \Lambda^{(0,1)}(N,T'N)$ satisfying the condition:

(5) $$\bar{\partial}\omega - \tfrac{1}{2}[\omega,\omega] = 0$$

such that it contains 0 and it cuts transversally the set

(32) $$\{\omega \circ g_\xi \ ; \ \xi \in C^\infty(N,T') \ , \ \xi \ \text{small}\} \ .$$

When we linearize the problem, we see by (29) that we are asked to find a complete set of representatives of the cohomology classes in the $T'N$ valued differential forms of type $(0,1)$. The standard way is to solve the equation

(33) $$\bar{\partial}\omega = 0 \ , \quad \bar{\partial}^*\omega = 0 \ .$$

This observation suggests that a good candidate for our C is the set of sufficiently small solutions of the equation

(34) $$\bar{\partial}\omega - \tfrac{1}{2}[\omega,\omega] = 0 \ , \quad \bar{\partial}^*\omega = 0 \ .$$

Actually it can be shown that such C forms the versal family.

The equation (34) is solved as follows: Since solutions of (34) satisfy the condition

(35) $\quad G\bar{\partial}^*(\bar{\partial}\omega - \frac{1}{2}[\omega,\omega]) + G\bar{\partial}\bar{\partial}^*\omega + H(\omega)$ is harmonic,

where G is the Green's operator and H is the harmonic projection, we first solve the equation (35) and decide which of the solutions of (35) are solutions of (34). Since the sufficiently small solutions of (35) form a finite dimensional complex manifold, it can be shown that solutions of (34) in this manifold form an analytic set. Now we can solve the equation (35) when we can invert the map

(36) $\quad \omega \to H(\omega) + G\bar{\partial}^*(\bar{\partial}\omega - \frac{1}{2}[\omega,\omega]) + G\bar{\partial}\bar{\partial}^*\omega = \omega - \frac{1}{2}G\bar{\partial}^*[\omega,\omega].$

Because of the ellipticity of the laplacian $\Delta = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$, the map (36) induces an analytic map of the Banach manifold obtained by completing with respect to Sobolov norm. Therefore to find the inverse of (36), we check that the differential at 0 is the indentity map theorem and apply Banach inverse mapping theorem.

To find CR analog of the above construction we merely have to replace the equation (5) by the equation (cf. (27))

(37) $\quad P(\varphi) = 0$

and the formula (29) by (31). Thus our problem is to invert the map

(38) $$\varphi \to H(\varphi) + N\bar{\partial}_b^* P(\varphi) + N\bar{\partial}_b \bar{\partial}_b^* \varphi$$

where N is the Neumann operator. (For subellipticity and Neumann operators, see the Kohn's article [4] in this volume.) However, the analogy fails here because the laplacian $\Delta_b = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*$ is not elliptic and hence the map (38) does not induce the map of a Sobolev Banach manifold into itself. The way to get around the difficulty is to note that Δ_b is subelliptic and use Nash-Moser inverse mapping theorem. The theorem says that, when we have a map like (38), if the differentials at points near 0 are subelliptic with uniform estimate and invertible then the map is invertible. However, the differentials of the map (38) at non zero points do not appear to be subelliptic. Thus we are forced to modify our construction: In order to obtain the sub-ellipticity of differentials we have to bring the boundary Cauchy Riemann operators of each ${}^{o}T''_{\varphi}$ into our picture. Now it is necessary to introduce a number of operators. Before proceeding further we note that the inverse in Nash Morse theorem is constructed by ingeniously combining Newton's algorithm and the smoothing operators (cf. [7]).

We recall that $\overline{\partial}_{E''}$- complex was constructed in (14) by means of the diagram (12) in the case $E'' = {}^{O}T''_{\varphi}$. However, we need such an operator for all sufficiently small φ (not merely for φ for which ${}^{O}T''_{\varphi}$ is a CR-structure). We obtain this by picking cross-sections of the vertical arrows in (12). Such cross-sections which are natural from our stand-point are induced by the decomposition (cf. (6))

$$\underline{C}TM = E'' + E' + F , \quad E'' = {}^{O}T''_{\varphi} .$$

We denote by $\overline{\partial}_b^{\varphi}$ the sequence of differential operators thus obtained:

$$\text{(39)} \qquad \overline{\partial}_b^{\varphi}\colon \Lambda_b^{(0,p)}(M,C) \to \Lambda_b^{(0,p+1)}(M,C) .$$

They form a complex if and only if ${}^{O}T''_{\varphi}$ is a CR-structure. In terms of a complex analytic chart $z = (z^1,\ldots,z^n)$ of N (and using the notations introduced near the end of §3),

$$\text{(40)} \qquad \overline{\partial}_b^{\varphi} f = \sum_{\ell} (Z_{\ell}^{\varphi} f)\overline{z^{\ell}} , \qquad f \in C^{\infty}(G,C)$$

$$\text{(41)} \qquad \overline{\partial}_b^{\varphi}\overline{z^{\ell}} = \Sigma_k \varphi^k \wedge \overline{\partial}_b^{\varphi}(h_k p^{\overline{\ell}}) .$$

The formulae (40) and (41) determine uniquely the operator $\overline{\partial}_b^{\varphi}$ because of the linearity and the rule $\overline{\partial}_b^{\varphi}(\theta \wedge \psi) = (\overline{\partial}_b^{\varphi}\theta) \wedge \psi + (-1)^{\ell}\theta \wedge \overline{\partial}_b^{\varphi}\psi$ where ℓ is the degree of θ .

It is interesting to note here the formula

$$\overline{\partial}_b^{\varphi} \circ \overline{\partial}_b^{\varphi} f = -\Sigma_k (\partial^{\tau} f / \partial z^k) P(\varphi)^k .$$

This could be used to show that $P(\varphi)$ is well-defined independent of choices in the defining formula (27). Since in our construction of versal families we work on $T'N|M$ valued differential forms of type $(0,p)_b$ we have to define $\overline{\partial}_b^{\varphi}$ for such forms. We do this by explicitly writing down the definition in terms of complex analytic chart z in N and showing that it is well-defined globally. For $\mu \in \Lambda_b^{(0,p)}(M, T'N|M)$ write

$$\mu = \Sigma_k \mu^k \otimes \partial/\partial z^k , \quad \mu^k \in \Lambda_b^{(0,p)}(G, \underline{C}) .$$

Then we define $\overline{\partial}_b^{\varphi}\mu$ on G by

$$\overline{\partial}_b^{\varphi}\mu = \Sigma_k (\overline{\partial}_b^{\varphi}\mu^k + \Sigma_{\ell} \Gamma_{\ell}^k(\varphi) \wedge \mu^{\ell}) \otimes \partial/\partial z^k$$

where

$$\Gamma_{\ell}^k(\varphi) = \Sigma_j ((\partial^{\tau}\varphi_{\overline{j}}^k / \partial z^{\ell}) z^{\overline{j}} - \varphi_{\overline{j}}^k \alpha_{\ell}^{\overline{j}}(\varphi)) ,$$

$$\alpha_{\ell}^{\overline{k}}(\varphi) = h_{\ell} \overline{\partial}_b^{\varphi} p^{\overline{k}} + \Sigma_j h_j \varphi^j \partial^{\tau} p^{\overline{k}} / \partial z^{\ell} .$$

By means of a hermitian metric we introduce $(\overline{\partial}_b^{\varphi})^*$. The laplacian $\Delta_b^{\varphi} = (\overline{\partial}_b^{\varphi})^* \overline{\partial}_b^{\varphi} + \overline{\partial}_b^{\varphi} (\overline{\partial}_b^{\varphi})^*$ is still subelliptic. The dimension of the kernel of Δ_b^{φ} may depend on φ. However, we can show that, for φ sufficiently small, the dimension of the sum of the eigen spaces of eigen values sufficiently small, say H'_φ, is independent of φ. Denote by ρ^φ the orthogonal projection (in L_2 norm) to H'_φ. Denote by N^φ the composition of $I - \rho^\varphi$ with Neumann operator of Δ_b^φ where I is the identity map. We have the formula:

$$\rho^\varphi + N^\varphi \Delta_b^\varphi \quad = \quad \text{the identity map} .$$

We use ρ^φ and N^φ instead of the harmonic projection and Neumann operator of Δ_b^φ because the latters do not depend smoothly on φ. When we write down $(\bar\partial_b^\varphi)^*$ in terms of a local chart, we find that partial derivatives of $\varphi_{\bar\ell}^k$ appear in the coefficients of the zero-th order terms of the expression. By a technical reason these coefficients cause some trouble in the construction of the universal family. Therefore we just take out the terms which contain partial derivatives of $\varphi_{\bar\ell}^k$ and piece them together by means of a partition of unity. In this way we construct $(\bar\partial_b^\varphi)^\#$. It is a differential operator having the same principal part as $\bar\partial_b^\varphi$.

We are ready to state what we will do instead of trying to find the inverse of the map (38). For each sufficiently small harmonic (with respect to Δ_b) $T'N|M$ valued differential form of type $(0,1)_b$, solve the equation

$$\rho^\varphi \varphi + N^\varphi((\bar\partial_b^\varphi)^* P(\varphi) + \bar\partial_b^\varphi(\bar\partial_b^\varphi)^* \varphi) = \rho^\varphi t .$$

It can be shown by means of Nash Moser theorem that the equation has a unique solution, say $\varphi(t)$, which is sufficiently small. Then we write down the equation for t so that ${}^{o}T''_{\varphi(t)}$ is a CR-structure. In this way we construct a family of CR-structures. By analyzing f_{ξ} closely we can show that the family induces the versal family of deformation of the isolated singularity we started with.

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